An Introduction to MUSIC and ESPRIT

GIRD Systems, Inc.
310 Terrace Ave.
Cincinnati, Ohio 45220

Based on

R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Trans. Antennas & Propagation*, vol. 34, no. 3, March 1986, and
R. Roy and T. Kailath, "ESPRIT – Estimation of signal parameters via rotation invariance techniques," *IEEE Trans. Acoust., Speech, Signal Proc.*, vol. 17, no. 7, July 1989

Introduction

- Well known high-resolution DOA algorithms
- Able to find DOAs of multiple sources
- High spatial resolution compared with other alg. (I.e., a few antennas can result in high accuracy)
- MUSIC stands for Multiple Signal Classifier
- ESPRIT stands for Estimation of Signal Parameters via Rotational Invariance Technique
- Apply to only narrowband signal sources

Narrowband Signal Sources

- A complex sinusoid

$$s(t) = \alpha e^{j\beta} e^{j\omega t} = \rho e^{j\omega t}$$

- A real sinusoid is a sum of two sinusoids

$$\alpha \cos(\omega t + \beta) = \frac{\alpha}{2} e^{j\beta} e^{j\omega t} + \frac{\alpha}{2} e^{-j\beta} e^{-j\omega t} = \rho_1 e^{j\omega t} + \rho_2 e^{-j\omega t}$$

- A delay of a sinusoid is a phase shift

$$s(t - t_0) = e^{-j\omega t_0} \rho e^{j\omega t} = e^{-j\omega t_0} s(t)$$

- Apply approximately to narrowband signals

Narrowband Signal Sources

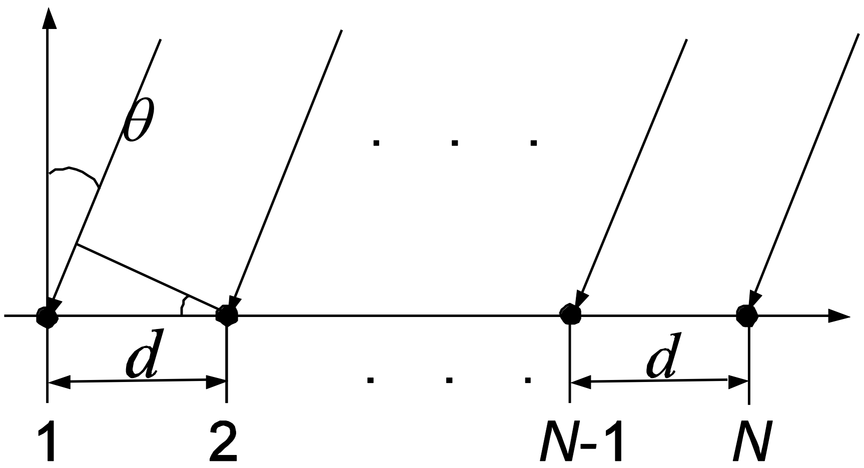
- Consider *I* narrowband signal sources

$$s_1(t) = \rho_1 e^{j\omega_1 t}, \quad s_2(t) = \rho_2 e^{j\omega_2 t}, \quad \cdots, \quad s_I(t) = \rho_I e^{j\omega_I t}$$

- Assume that all frequencies are different
- Assume that all amplitudes are uncorrelated

$$E\{\rho_i \rho_j\} = \begin{cases} \sigma_i^2; & i = j \\ 0; & i \neq j \end{cases}$$

A Uniform Linear Array



A signal source $s(t) = \rho e^{j\omega t}$ "impinges" on the array with an angle θ

c: propagation speed

- If the received signal at sensor 1 is $x_1(t) = s(t)$
- Then it is delayed at sensor *i* by $\Delta_i = \dfrac{(i-1)d\sin\theta}{c}$
- Then the received signal at sensor *i* is

$$x_i(t) = e^{-j\omega\Delta_i} s_1(t) = e^{-j\omega\Delta_i} s(t) = e^{-j\omega\frac{(i-1)d\sin\theta}{c}} s(t)$$

Signal Model

- Put received signals at all N sensors together:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_N(t) \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-j\omega\frac{d\sin\theta}{c}} \\ e^{-j\omega\frac{2d\sin\theta}{c}} \\ \vdots \\ e^{-j\omega\frac{(N-1)d\sin\theta}{c}} \end{bmatrix} s(t) = \mathbf{a}(\theta)s(t)$$

- $\mathbf{a}(\theta)$ is called a "steering vector"

Signal Model

- If there are I sources signals received by the array, we get a "signal model":

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t)$$

$\mathbf{x}(t)$ --- received signal vector (N by 1)
$\mathbf{s}(t)$ --- source signal vector (I by 1)
$\mathbf{n}(t)$ --- noise vector (N by 1)
$\mathbf{A} = [\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_I)]$ (N by I)
$\mathbf{s}(t) = [s_1(t), \cdots, s_I(t)]^T$

- Sources are independent, noises are uncorrelated
- Column of **A** can also be normalized

The MUSIC Algorithm

- Compute the N x N correlation matrix

$$\mathbf{R}_{\mathbf{x}} = E\{\mathbf{x}(t)\mathbf{x}^{H}(t)\} = \mathbf{A}\mathbf{R}_{\mathbf{s}}\mathbf{A}^{H} + \sigma_0^2\mathbf{I}$$

where $$\mathbf{R}_{\mathbf{s}} = E\{\mathbf{s}(t)\mathbf{s}^{H}(t)\} = diag.\{\sigma_1^2, \cdots, \sigma_I^2\}$$

- If the sources are somewhat correlated so $\mathbf{R}_{\mathbf{s}}$ is not diagonal, it will still work if $\mathbf{R}_{\mathbf{s}}$ has full rank.
- If the sources are correlated such that $\mathbf{R}_{\mathbf{s}}$ is rank deficient, then it is a problem. A common solution is "spatial smoothing".

Q: Why is the rank of $\mathbf{R}_{\mathbf{s}}$ (being I) so important?

A: It defines the dimension of the signal subspace.

The MUSIC Algorithm

- For $N > I$, the matrix $\mathbf{A}\mathbf{R}_{\mathbf{s}}\mathbf{A}^H$ is singular, i.e.,

$$\det[\mathbf{A}\mathbf{R}_{\mathbf{s}}\mathbf{A}^H] = \det[\mathbf{R}_{\mathbf{x}} - \sigma_0^2\mathbf{I}] = 0$$

- But this implies that σ_0^2 is an eigenvalue of $\mathbf{R}_{\mathbf{x}}$
- Since the dimension of the null space of $\mathbf{A}\mathbf{R}_{\mathbf{s}}\mathbf{A}^H$ is *N*-*I*, there are *N*-*I* such eigenvalues σ_0^2 of $\mathbf{R}_{\mathbf{x}}$
- Since both $\mathbf{R}_{\mathbf{x}}$ and $\mathbf{A}\mathbf{R}_{\mathbf{s}}\mathbf{A}^H$ are non-negative definite, there are *I* other eigenvalues σ_i^2 such that $\sigma_i^2 > \sigma_0^2 > 0$
- Let $\mathbf{u}_i$ be the *i*th eigenvector of $\mathbf{R}_{\mathbf{x}}$ corresponding to σ_i^2

$$\mathbf{R}_{\mathbf{x}}\mathbf{u}_i = [\mathbf{A}\mathbf{R}_{\mathbf{s}}\mathbf{A}^H + \sigma_0^2\mathbf{I}]\mathbf{u}_i = \sigma_i^2\mathbf{u}_i; \quad i = 1, 2, \cdots, N$$

$$\sigma_i^2 > \sigma_0^2 > 0,\ i = 1, \cdots, I; \quad \sigma_i^2 = \sigma_0^2,\ i = I+1, \cdots, N$$

The MUSIC Algorithm

$$\mathbf{R_x u}_i = [\mathbf{AR_s A}^H + \sigma_0^2 \mathbf{I}]\mathbf{u}_i = \sigma_i^2 \mathbf{u}_i; \quad i = 1, 2, \cdots, N$$

- This implies

$$\mathbf{AR_s A}^H \mathbf{u}_i = (\sigma_i^2 - \sigma_0^2)\mathbf{u}_i; \quad i = 1, 2, \cdots, N$$

$$\mathbf{AR_s A}^H \mathbf{u}_i = \begin{cases} (\sigma_i^2 - \sigma_0^2)\mathbf{u}_i; & i = 1, 2, \cdots, I \\ 0; & i = I+1, \cdots, N \end{cases}$$

- Partition the *N*-dimensional vector space into the signal subspace $\mathbf{U_s}$ and the noise subspace $\mathbf{U_n}$

$$[\mathbf{U_s} \quad \mathbf{U_n}] = [\underbrace{\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_I}_{\mathbf{U_s}:(\sigma_i^2 - \sigma_0^2) > 0 \text{ eigenvalues}} \quad \underbrace{\mathbf{u}_{I+1} \quad \cdots \quad \mathbf{u}_N}_{\mathbf{U_n}: 0 \text{ eigenvalues}}]$$

The MUSIC Algorithm

- The steering vector $\mathbf{a}(\theta_i)$ is in the signal subspace
- Signal subspace is orthogonal to noise subspace

$$\mathbf{A}\mathbf{R}_{\mathbf{s}}\mathbf{A}^{H}\mathbf{u}_i = \begin{cases} (\sigma_i^2 - \sigma_0^2)\mathbf{u}_i; & i = 1, 2, \cdots, I \quad (1) \\ 0; & i = I+1, \cdots, N \quad (2) \end{cases}$$

(1) means I linear combinations of columns of **A** equal the signal subspace spanned by columns of $\mathbf{U_s}$

(2) means the linear combinations of columns of **A**, i.e., the signal subspace, is orthogonal to $\mathbf{U_n}$

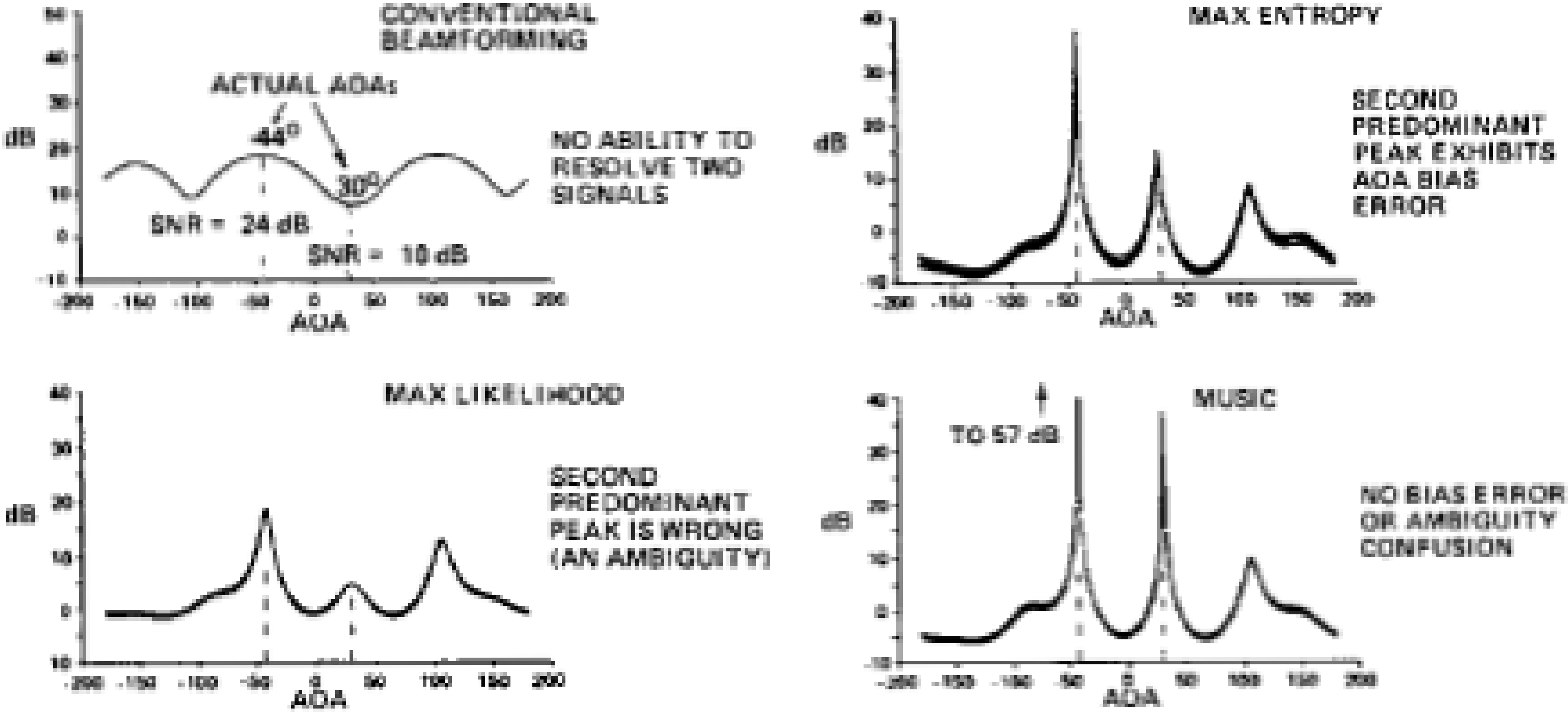
The MUSIC Algorithm

- The steering vector $\mathbf{a}(\theta_i)$ is in the signal subspace
- Signal subspace is orthogonal to noise subspace
- This implies that $\mathbf{a}^H(\theta_i)\mathbf{U_n} = \mathbf{0}$
- So the MUSIC algorithm searches through all angles θ, and plots the "spatial spectrum"

$$P(\theta) = \frac{1}{\mathbf{a}^H(\theta)\mathbf{U_n}}$$

- Wherever $\theta = \theta_i$, $P(\theta)$ exhibits a peak
- Peak detection will give spatial angles of all incident sources

The MUSIC Algorithm



MUSIC spatial spectrum compared with other methods

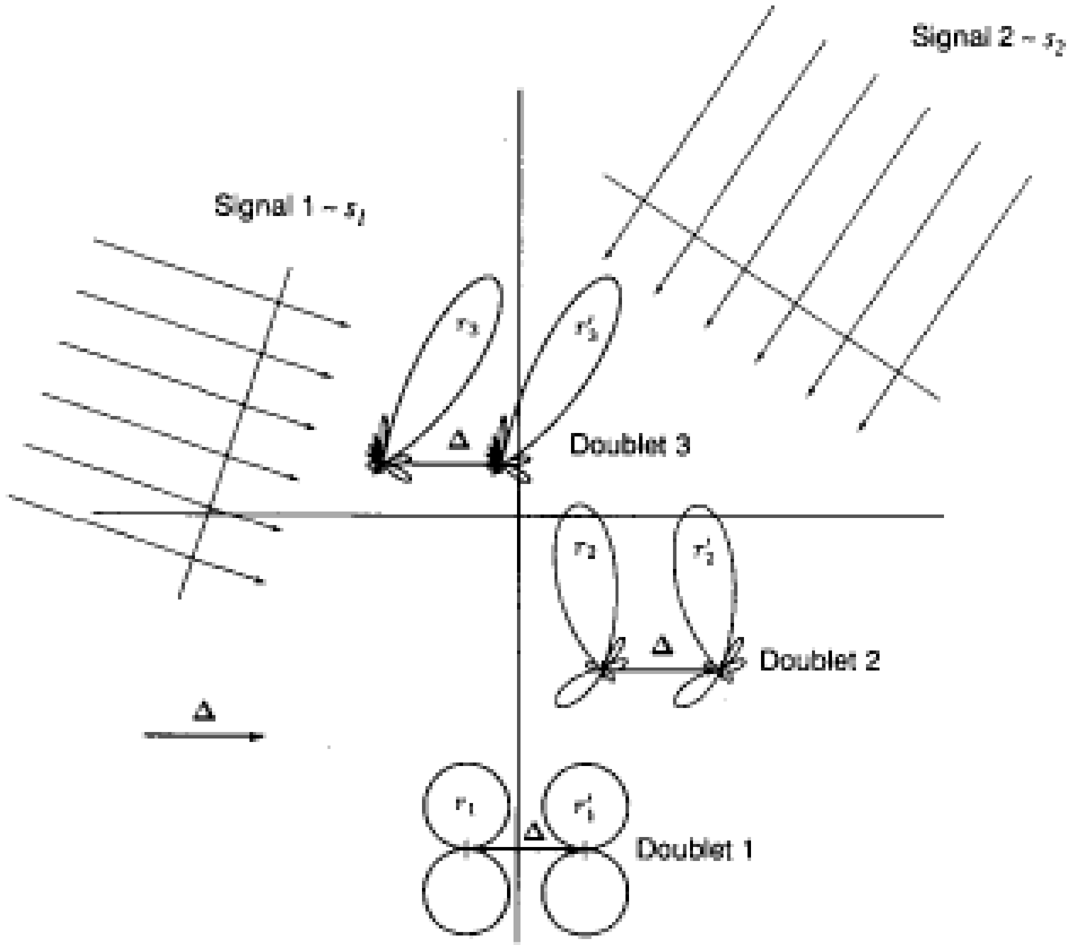
Pros/Cons of The MUSIC Algorithm

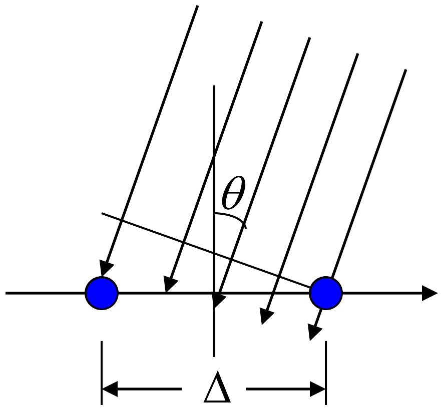
- Works for other array shapes, need to know sensor positions
- Very sensitive to sensor position, gain, and phase errors, need careful calibration to make it work well
- Searching through all θ could be computationally expensive
- The ESPRIT algorithm overcomes such shortcomings to some degree
- ESPRIT relaxes the calibration task somewhat
- ESPRIT takes much less computation
- But ESPRIT takes twice as many sensors

The ESPRIT Algorithm

- Based on “doublets” of sensors, i.e., in each pair of sensors the two should be identical, and all doubles should line up completely in the same direction with a displacement vector $\boldsymbol{\Delta}$ having magnitude Δ
- Otherwise there are no restrictions, the sensor patterns could be very different from one pair to another
- The positions of the doublets are also arbitrary
- This makes calibration a little easier
- Assume N sets of doublets, i.e., $2N$ sensors
- Assume I sources, $N > I$

The ESPRIT Algorithm





The amount of delay between two sensors in each doublet for a given incident signal is the same for all doublets, which is

$$\Delta \sin\theta / c$$

A sensor array of doublets

The ESPRIT Algorithm

- This array is consisted of two identical subarrays, $\mathbf{Z_x}$ and $\mathbf{Z_y}$, displaced from each other by $\mathbf{\Delta}$

$$\mathbf{x}(t) = \sum_{i=1}^{I} \mathbf{a}(\theta_i) s_i(t) + \mathbf{n_x}(t) = \mathbf{As}(t) + \mathbf{n_x}(t)$$

$$\mathbf{y}(t) = \sum_{i=1}^{I} \mathbf{a}(\theta_i) e^{j\gamma_i} s_i(t) + \mathbf{n_x}(t) = \mathbf{A\Phi s}(t) + \mathbf{n_x}(t)$$

$$\gamma_i = \omega_0 \Delta \sin\theta_i / c \qquad \mathbf{\Phi} = diag.\{e^{j\gamma_1}, e^{j\gamma_2}, \cdots, e^{j\gamma_I}\}$$

- The steering vector $\mathbf{a}(\theta)$ depends on the array geometry, and should be known just like in MUSIC

The ESPRIT Algorithm

- The objective is to estimate $\mathbf{\Phi}$, thereby obtaining θ_i
- Define

$$\mathbf{z}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ \mathbf{A\Phi} \end{bmatrix} \mathbf{s}(t) + \begin{bmatrix} \mathbf{n_x}(t) \\ \mathbf{n_y}(t) \end{bmatrix} = \bar{\mathbf{A}}\mathbf{s}(t) + \mathbf{n_z}(t)$$

- Compute the 2*N* x 2*N* correlation matrix

$$\mathbf{R_z} = E\{\mathbf{z}(t)\mathbf{z}^H(t)\} = \bar{\mathbf{A}}\mathbf{R_s}\bar{\mathbf{A}}^H + \sigma_0^2\mathbf{I}$$

- Since there are *I* sources, the *I* eigenvectors of $\mathbf{R_z}$ corresponding to the *I* largest eigenvalues form the signal subspace $\mathbf{U_s}$; The remaining 2*N-I* eigenvectors form the noise subspace $\mathbf{U_n}$

The ESPRIT Algorithm

- $\mathbf{U}_s$ is 2N x I, and its span is the same as the span of $\bar{\mathbf{A}}$
- Therefore, there exists a unique nonsingular I x I matrix $\mathbf{T}$ such that (**A** needs to be known here)

$$\mathbf{U}_s = \bar{\mathbf{A}}\mathbf{T}$$

- Partition $\mathbf{U}_s$ into two N x I submatrices

$$\mathbf{U}_s = \begin{bmatrix} \mathbf{U}_x \\ \mathbf{U}_y \end{bmatrix} = \begin{bmatrix} \mathbf{AT} \\ \mathbf{A\Phi T} \end{bmatrix}$$

- The columns of both $\mathbf{U}_x$ and $\mathbf{U}_y$ are linear combinations of $\mathbf{A}$, so each of them has a column rank I

The ESPRIT Algorithm

- Define an *N* x 2*I* matrix, which has rank *I*

$$\mathbf{U_{xy}} = \begin{bmatrix} \mathbf{U_x} & \mathbf{U_y} \end{bmatrix}$$

- Therefore, $\mathbf{U_{xy}}$ has a null space with dimension *I*, i.e., there exists a 2*I* x *I* matrix **F** such that

$$\mathbf{U_{xy}F} = \mathbf{0} \Leftrightarrow \begin{bmatrix} \mathbf{U_x} & \mathbf{U_y} \end{bmatrix} \begin{bmatrix} \mathbf{F_x} \\ \mathbf{F_y} \end{bmatrix} = \mathbf{U_x F_x} + \mathbf{U_y F_y} = \mathbf{0}$$

$$\Leftrightarrow \mathbf{ATF_x} + \mathbf{A\Phi TF_y} = \mathbf{0} \Leftrightarrow \mathbf{A\Phi TF_y} = -\mathbf{ATF_x}$$

- Then the above gives, since **T** has full column rank

$$\mathbf{A\Phi T} = -\mathbf{ATF_x F_y^{-1}} \Leftrightarrow \mathbf{A\Phi} = \mathbf{ATF_x F_y^{-1} T}^{-1} \Leftrightarrow \mathbf{\Phi} = \mathbf{TF_x F_y^{-1} T}^{-1}$$

The ESPRIT Algorithm

- The final algorithm is

$$\mathbf{\Phi} = \mathbf{T}\mathbf{F}_{\mathbf{x}}\mathbf{F}_{\mathbf{y}}^{-1}\mathbf{T}^{-1}$$

- In practice, the measurement could be noisy, and there could be array calibration errors, so a total least squares ESPRIT is used
- The estimation is "one shot", even with matrix inversions the computation is much less than the MUSIC search
- We must have $N > I$ for ESPRIT to work with $2N$ sensors, so need twice as many sensors as MUSIC

The ESPRIT Algorithm

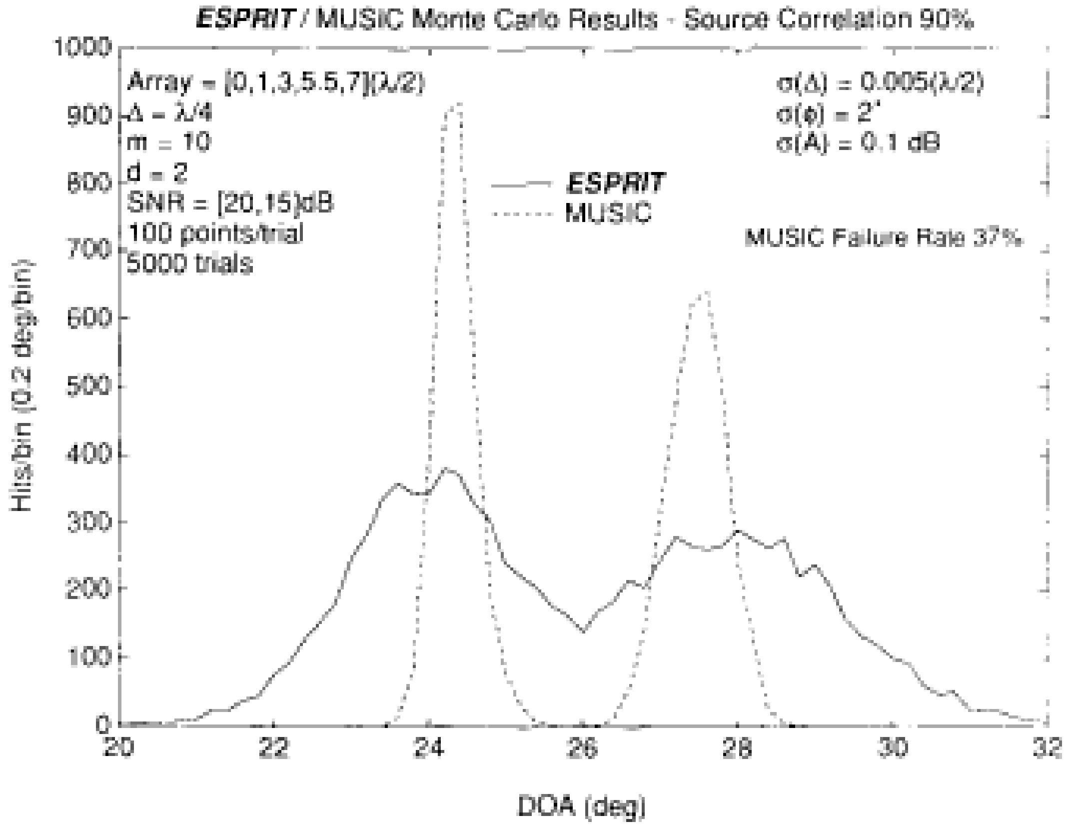


Fig. 4. Histogram of MUSIC and ESPRIT results—random 10-element linear array, source correlation 90 percent, small array aperture ($\Delta = \lambda/4$).

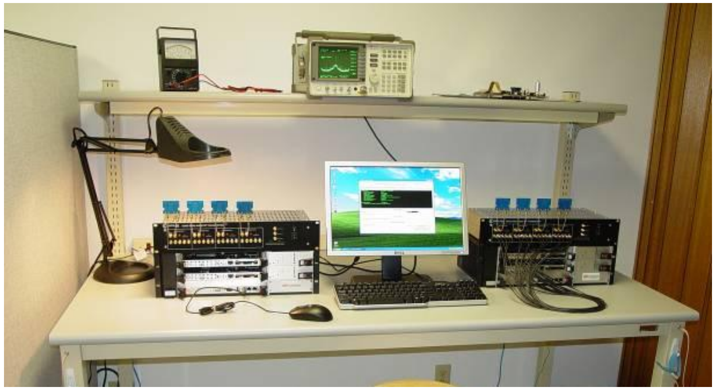
Simulation results of ESPRIT comparing with MUSIC

Conclusions

- Both methods are high-resolution, much better spatial resolution than beamforming and other methods
- Both methods are able to detect multiple sources
- If sensors are expensive and few, and if computation is not of concern, MUSIC is suitable
- If there are plenty of sensors compared with the number of sources to detect, and if computational power is limited, ESPRIT is suitable

About GIRD Systems, Inc.

- Founded in 2000 - based in Cincinnati
- Specializes in communications and signal processing, especially developing novel algorithms to solve challenging problems
- As of Nov. 2009, won more than 13 Phase I awards and 6 Phase II awards from Navy, Air Force, Army
- Partnerships with many large contractors including Northrop Grumman, L-3 Communications, etc.



About GIRD Systems, Inc.

- Key Technology Areas
  - Interference Mitigation (no reference, in-band)
  - Direction Finding (wideband, high-resolution)
  - Location/Navigation (Assisted GPS, GPS denied, signals of opportunity)
  - Wireless Network Security (physical layer)
  - Power Amplifier Linearization
  - Novel Communications systems/modeling
- Contact Information:

GIRD Systems, Inc.
310 Terrace Ave.
Cincinnati, OH 45220
USA

Phone: (513) 281-2900
Fax: (513) 281-3494
E-mail: info@girdsystems.com
Web: www.girdsystems.com